BEAM DESIGN FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

DESIGN AID No. 6
BEAM FORMULAS WITH SHEAR AND MOMENT DIAGRAMS

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Introduction

Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the *Western Woods Use Book*, 4th edition, and are provided herein as a courtesy of [Western Wood Products Association](https://www.westernwoodproducts.org).

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Figure 1  Simple Beam – Uniformly Distributed Load

\[ R = V \quad \ldots \ldots \ldots \ldots = \frac{w \ell}{2} \]
\[ V_x \quad \ldots \ldots \ldots \ldots = \frac{w}{2} \left( \frac{\ell}{2} - x \right) \]
\[ M_{\text{max}} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{w \ell^2}{8} \]
\[ M_x \quad \ldots \ldots \ldots \ldots = \frac{wx}{2} (\ell - x) \]
\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \ldots \ldots = \frac{5w\ell^4}{384EI} \]
\[ \Delta_x \quad \ldots \ldots \ldots \ldots = \frac{wx}{24EI} (\ell^3 - 2\ell x^2 + x^3) \]

Figure 2  Simple Beam – Uniform Load Partially Distributed

\[ R_1 = V_1 \quad \text{(max when } a < c) \quad \ldots \ldots = \frac{wb}{2\ell} (2c + b) \]
\[ R_2 = V_2 \quad \text{(max when } a > c) \quad \ldots \ldots = \frac{wb}{2\ell} (2a + b) \]
\[ V_x \quad \text{(when } x > a \text{ and } < (a + b)) \quad \ldots \ldots = R_1 - w(x - a) \]
\[ M_{\text{max}} \quad \text{at } x = a + \frac{R_1}{w} \quad \ldots \ldots = R_1 \left( a + \frac{R_1}{2w} \right) \]
\[ M_x \quad \text{(when } x < a) \quad \ldots \ldots \ldots = R_1 x \]
\[ M_x \quad \text{(when } x > a \text{ and } < (a + b)) \quad \ldots \ldots = R_1 x - \frac{w}{2}(x - a)^2 \]
\[ M_x \quad \text{(when } x > (a + b)) \quad \ldots \ldots = R_2 (\ell - x) \]
Figure 3  
**Simple Beam – Uniform Load Partially Distributed at One End**

\[ R_1 = V_1 \quad \dot{\dot{\ldots}} \quad = \frac{wa}{2\ell} (2\ell - a) \]

\[ R_2 = V_2 \quad \dot{\ldots} \quad = \frac{wa^2}{2\ell} \]

\[ V_x \text{ (when } x < a \text{)} \quad \dot{\ldots} \quad = R_1 - wx \]

\[ M_{\text{max}} \text{ at } x = \frac{R_1}{w} \quad \dot{\ldots} \quad = \frac{R_1^2}{2w} \]

\[ M_x \text{ (when } x < a \text{)} \quad \dot{\ldots} \quad = R_1x - \frac{wx^2}{2} \]

\[ M_x \text{ (when } x > a \text{)} \quad \dot{\ldots} \quad = R_2(\ell - x) \]

\[ \Delta_x \text{ (when } x < a \text{)} \quad \dot{\ldots} \quad = \frac{wx}{24 E I \ell} \left( a^2(2\ell - a)^2 - 2ax(2\ell - a) + \ell x^3 \right) \]

\[ \Delta_x \text{ (when } x > a \text{)} \quad \dot{\ldots} \quad = \frac{wa^2(\ell - x)}{24 E I \ell} (4\ell x - 2x^2 - a^2) \]

Figure 4  
**Simple Beam – Uniform Load Partially Distributed at Each End**

\[ R_1 = V_1 \quad \dot{\ldots} \quad = \frac{w_1a(2\ell - a) + w_2c^2}{2\ell} \]

\[ R_2 = V_2 \quad \dot{\ldots} \quad = \frac{w_2c(2\ell - c) + w_1a^2}{2\ell} \]

\[ V_x \text{ (when } x < a \text{)} \quad \dot{\ldots} \quad = R_1 - w_1x \]

\[ V_x \text{ (when } x > a \text{ and } (a + b) \text{)} \quad \dot{\ldots} \quad = R_1 - w_1a \]

\[ V_x \text{ (when } x > (a + b) \text{)} \quad \dot{\ldots} \quad = R_2 - w_2(\ell - x) \]

\[ M_{\text{max}} \text{ at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1a \quad \dot{\ldots} \quad = \frac{R_1^2}{2w_1} \]

\[ M_{\text{max}} \text{ at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2c \quad \dot{\ldots} \quad = \frac{R_2^2}{2w_2} \]

\[ M_x \text{ (when } x < a \text{)} \quad \dot{\ldots} \quad = R_1x - \frac{wx^2}{2} \]

\[ M_x \text{ (when } x > a \text{ and } (a + b) \text{)} \quad \dot{\ldots} \quad = R_1x - \frac{w_1a}{2} (2x - a) \]

\[ M_x \text{ (when } x > (a + b) \text{)} \quad \dot{\ldots} \quad = R_2(\ell - x) - \frac{w_2(\ell - x)^2}{2} \]
Figure 5  Simple Beam – Load Increasing Uniformly to One End

\[ R_1 = V_1 = \frac{W}{3} \]

\[ R_2 = V_2 = \frac{2W}{3} \]

\[ V_x = \frac{W}{\frac{3}{2}} - \frac{Wx^2}{\ell^2} \]

\[ M_{\text{max}} \left( \text{at } x = \frac{\ell}{\sqrt{3}} = 0.5774\ell \right) = \frac{2W\ell}{9\sqrt{3}} = 0.1283W\ell \]

\[ M_x = \frac{Wx}{3\ell^2} (\ell^2 - x^2) \]

\[ \Delta_{\text{max}} \left( \text{at } x = \ell \left( 1 - \frac{8}{15} \right) = 0.5193\ell \right) = \frac{0.01304W\ell^3}{EI} \]

\[ \Delta_x = \frac{Wx}{180EI\ell^2} (3x^4 - 10\ell^2x^2 + 7\ell^4) \]

Figure 6  Simple Beam – Load Increasing Uniformly to Center

\[ R = V = \frac{W}{2} \]

\[ V_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{W}{2\ell^2} (\ell^2 - 4x^2) \]

\[ M_{\text{max}} \left( \text{at center} \right) = \frac{W\ell}{6} \]

\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Wx}{2\ell^2} \left( 1 - \frac{2x^2}{3\ell^2} \right) \]

\[ \Delta_{\text{max}} \left( \text{at center} \right) = \frac{W\ell^3}{60EI} \]

\[ \Delta_x = \frac{Wx}{480EI\ell^2} (5\ell^2 - 4x^2)^2 \]
Figure 7  Simple Beam – Concentrated Load at Center

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{P\ell}{4} \]

\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{Px}{2} \]

\[ \Delta_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{P\ell^3}{48EI} \]

\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \ldots \ldots = \frac{Px}{48EI} \left(3\ell^2 - 4x^2\right) \]

Figure 8  Simple Beam – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots \ldots = \frac{Pb}{\ell} \]

\[ R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots \ldots = \frac{Pa}{\ell} \]

\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pab}{\ell} \]

\[ M_x \left( \text{when } x < a \right) \quad \ldots \ldots \ldots = \frac{Pbx}{\ell} \]

\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \quad = \frac{Pab(a + 2b)^{\frac{3}{2}}(a + 2b)}{27EI\ell} \]

\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pa^2b^2}{3EI\ell} \]

\[ \Delta_x \left( \text{when } x < a \right) \quad \ldots \ldots \ldots = \frac{Pbx}{6EI\ell} \left(\ell^2 - b^2 - x^2\right) \]

\[ \Delta_x \left( \text{when } x > a \right) \quad \ldots \ldots \ldots = \frac{Pa(x - b)}{6EI\ell} \left(2\ell x - x^2 - a^2\right) \]
**Figure 9** Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

\[ R = V \quad \ldots \quad = P \]
\[ M_{\text{max}} \text{ (between loads)} \quad \ldots \quad = Pa \]
\[ M_s \text{ (when } x < a) \quad \ldots \quad = Px \]
\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \quad = \frac{Pa}{24EI} (3\ell^2 - 4a^2) \]
\[ \Delta_s \text{ (when } x < a) \quad \ldots \quad = \frac{Px}{6EI} (3\ell a - 3a^2 - x^2) \]
\[ \Delta_s \text{ (when } x > a \text{ and } < (\ell - a)) \quad \ldots \quad = \frac{Pa}{6EI} (3\ell x - 3x^2 - a^2) \]

**Figure 10** Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 \text{ (max when } a < b) \quad \ldots \quad = \frac{P}{\ell} (\ell - a + b) \]
\[ R_2 = V_2 \text{ (max when } a > b) \quad \ldots \quad = \frac{P}{\ell} (\ell - b + a) \]
\[ V_s \text{ (when } x > a \text{ and } < (\ell - b)) \quad \ldots \quad = \frac{P}{\ell} (b - a) \]
\[ M_s \text{ (max when } a > b) \quad \ldots \quad = R_1 a \]
\[ M_2 \text{ (max when } a < b) \quad \ldots \quad = R_2 b \]
\[ M_s \text{ (when } x < a) \quad \ldots \quad = R_s x \]
\[ M_s \text{ (when } x > a \text{ and } < (\ell - b)) \quad \ldots \quad = R_s x - P(x - a) \]
**Figure 11** Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed

\[
R_1 = V_1 = \frac{P_1(\ell - a) + P_2b}{\ell}
\]

\[
R_2 = V_2 = \frac{P_1a + P_2(\ell - b)}{\ell}
\]

\[
V_x \quad \text{(when } x > a \text{ and } < (\ell - b)) \quad \text{. . . } = R_1 - P_1
\]

\[
M_x \quad \text{(max when } R_1 < P_1) \quad \text{. . . } = R_1 a
\]

\[
M_x \quad \text{(max when } R_2 < P_2) \quad \text{. . . } = R_2 b
\]

\[
M_x \quad \text{(when } x < a) \quad \text{. . . } = R_1 x
\]

\[
M_x \quad \text{(when } x > a \text{ and } < (\ell - b)) \quad \text{. . . } = R_1 x - P_1(x - a)
\]

**Figure 12** Cantilever Beam – Uniformly Distributed Load

\[
R = V = \frac{w \ell}{2}
\]

\[
V_x = \frac{w x}{2}
\]

\[
M_{max} \quad \text{(at fixed end)} \quad \text{. . . } = \frac{w \ell^3}{2}
\]

\[
M_x = \frac{w x^3}{2}
\]

\[
\Delta_{max} \quad \text{(at free end)} \quad \text{. . . } = \frac{w \ell^4}{8EI}
\]

\[
\Delta_x = \frac{w}{24EI} (x^4 - 4x^3 + 3x^4)
\]
Figure 13  Cantilever Beam – Concentrated Load at Free End

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots = P\ell \]
\[ M_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = P\ell x \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots = \frac{P\ell^2}{3EI} \]
\[ \Delta_x \quad \ldots \ldots \ldots \ldots \ldots \ldots = \frac{P}{6EI} (2\ell^3 - 3\ell^2x + x^3) \]

Figure 14  Cantilever Beam – Concentrated Load at Any Point

\[ R = V \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (at fixed end)} \quad \ldots \ldots \ldots = Pb \]
\[ M_x \text{ (when } x > a) \quad \ldots \ldots \ldots = P(x - a) \]
\[ \Delta_{\text{max}} \text{ (at free end)} \quad \ldots \ldots \ldots = \frac{Pb^2}{6EI} (3\ell - b) \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \ldots \ldots = \frac{Pb^3}{3EI} \]
\[ \Delta_x \text{ (when } x < a) \quad \ldots \ldots \ldots = \frac{Pb^2}{6EI} (3\ell - 3x - b) \]
\[ \Delta_x \text{ (when } x > a) \quad \ldots \ldots \ldots = \frac{P(\ell - x)^2}{6EI} (3b - \ell + x) \]
Figure 15  Beam Fixed at One End, Supported at Other – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{3w\ell}{8} \]

\[ R_2 = V_2 = \frac{5w\ell}{8} \]

\[ V_x = R_1 - wx \]

\[ M_{\text{max}} = \frac{w\ell^2}{8} \]

\[ M_1 \left( \text{at } x = \frac{3}{8} \ell \right) = \frac{9}{128} w\ell^2 \]

\[ M_x = R_1 x - \frac{wx^2}{2} \]

\[ \Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{16}(1 + \sqrt{33}) = .4215 \ell \right) = \frac{w\ell^4}{185EI} \]

\[ \Delta_x = \frac{wx}{48EI} (\ell^3 - 3\ell x^2 + 2x^3) \]

Figure 16  Beam Fixed at One End, Supported at Other – Concentrated Load at Center

\[ R_1 = V_1 = \frac{5P}{16} \]

\[ R_2 = V_2 = \frac{11P}{16} \]

\[ M_{\text{max}} \left( \text{at fixed end} \right) = \frac{3P\ell}{16} \]

\[ M_1 \left( \text{at point of load} \right) = \frac{5P\ell}{32} \]

\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{5Px}{16} \]

\[ M_x \left( \text{when } x > \frac{\ell}{2} \right) = P \left( \frac{\ell}{2} - \frac{11x}{16} \right) \]

\[ \Delta_{\text{max}} \left( \text{at } x = \frac{\ell}{15} = .4472\ell \right) = \frac{P\ell^3}{48EI\sqrt{5}} = .009317 \frac{P\ell^3}{EI} \]

\[ \Delta_x \left( \text{at point of load} \right) = \frac{7P\ell^5}{768EI} \]

\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Px}{96EI} (3\ell^2 - 5x^2) \]

\[ \Delta_x \left( \text{when } x > \frac{\ell}{2} \right) = \frac{P}{96EI} (x - \ell)^2(11x - 2\ell) \]
Figure 17  Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point

\[ R_1 = V_1 \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pb}{2\ell} (a + 2\ell) \]

\[ R_2 = V_2 \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pa}{2\ell} (3\ell^2 - a^2) \]

\[ M_1 \text{ (at point of load)} \quad \ldots \quad \ldots \quad \ldots \quad = R_1 a \]

\[ M_1 \text{ (at fixed end)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pab}{2\ell^2} (a + \ell) \]

\[ M_3 \text{ (when } x < a) \quad \ldots \quad \ldots \quad \ldots \quad = R_1 x \]

\[ M_3 \text{ (when } x > a) \quad \ldots \quad \ldots \quad \ldots \quad = R_1 x - P(x - a) \]

\[ \Delta_{\text{max}} \left( \text{when } a < 0.414\ell \text{ at } x = \ell (\ell^2 + a^2) \right) = \frac{Pa}{3E\ell} (\ell^2 - a^2)^3 \]

\[ \Delta_{\text{max}} \left( \text{when } a > 0.414\ell \text{ at } x = \ell \sqrt{a^2 + 2a + 2a} \right) = \frac{Pab}{6E\ell} \sqrt{a^2 + 2a + 2a} \]

\[ \Delta_x \text{ (at point of load)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pa}{12E\ell^3} (3\ell + a) \]

\[ \Delta_x \text{ (when } x < a) \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pb^2 x}{12E\ell^4} (3a\ell^2 - 2ax^2 - ax^2) \]

\[ \Delta_x \text{ (when } x > a) \quad \ldots \quad \ldots \quad \ldots \quad = \frac{Pa}{12E\ell^4} (2a - x)^2 (\ell^2 - a^2 - 2a^2 \ell) \]

Figure 18  Beam Overhanging One Support – Uniformly Distributed Load

\[ R_1 = V_1 \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega}{2\ell} (\ell^2 - a^2) \]

\[ R_2 = V_2 + V_1 \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega}{2\ell} (\ell + a)^2 \]

\[ V_2 \quad \ldots \quad \ldots \quad \ldots \quad = \omega a \]

\[ V_3 \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega}{2\ell} (\ell^2 + a^2) \]

\[ V_x \text{ (between supports)} \quad \ldots \quad \ldots \quad \ldots \quad = R_1 - \omega x \]

\[ V_{x_1} \text{ (for overhang)} \quad \ldots \quad \ldots \quad \ldots \quad = \omega (a - x_1) \]

\[ M_1 \left( \text{at } x = \ell \left[ \frac{1}{2} - \frac{a^2}{\ell^2} \right] \right) \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega a^2}{8\ell^3} (\ell + a)^2 (\ell - a)^2 \]

\[ M_2 \text{ (at } R_2) \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega a^2}{2} \]

\[ M_3 \text{ (between supports)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega x}{2\ell} (\ell^2 - a^2 - x\ell) \]

\[ M_{x_1} \text{ (for overhang)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega}{2} (a - x_1)^2 \]

\[ \Delta_x \text{ (between supports)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega x}{24E\ell^4} (\ell^4 - 2 \ell^2 x^2 + \ell x^3 - 2a^2 \ell^2 + 2a^2 x^2) \]

\[ \Delta_{x_1} \text{ (for overhang)} \quad \ldots \quad \ldots \quad \ldots \quad = \frac{\omega x}{24E\ell} (4a^2 \ell - \ell^3 + 6a^2 x_1 - 4ax_1^2 + x_1) \]
Figure 19  Beam Overhanging One Support – Uniformly Distributed Load on Overhang

\[
R_1 = V_1 = \frac{wa^2}{2\ell} \\
R_2 = V_1 + V_2 = \frac{wa^2}{2\ell}(2\ell + a) \\
V_2 = wa \\
V_{x_1} \text{ (for overhang)} = wa(a - x_1) \\
M_{\text{max}} \text{ (at } R_1) = \frac{wa^2}{2} \\
M_s \text{ (between supports)} = \frac{wa^2x}{2\ell} \\
M_{x_1} \text{ (for overhang)} = \frac{w}{2}(a - x_1)^2 \\
\Delta_{\text{max}} \text{ (between supports at } x = \frac{\ell}{\sqrt{3}}) = \frac{wa^2\ell^3}{18\sqrt{3}EI} = 0.03208 \frac{wa^2\ell^3}{EI} \\
\Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{wa^2}{24EI} (4\ell + 3a) \\
\Delta_s \text{ (between supports)} = \frac{wa^2}{12EI}(\ell^2 - x^2) \\
\Delta_{x_1} \text{ (for overhang)} = \frac{wx_1}{24EI} (4a^2\ell + 6a^2x_1 - 4ax_1^2 + x_1^2)
\]

Figure 20  Beam Overhanging One Support – Concentrated Load at End of Overhang

\[
R_1 = V_1 = \frac{Pa}{\ell} \\
R_2 = V_1 + V_2 = \frac{P}{\ell}(\ell + a) \\
V_2 = P \\
M_{\text{max}} \text{ (at } R_1) = Pa \\
M_s \text{ (between supports)} = \frac{Pax}{\ell} \\
M_{x_1} \text{ (for overhang)} = P(a - x_1) \\
\Delta_{\text{max}} \text{ (between supports at } x = \frac{\ell}{\sqrt{3}}) = \frac{Pa\ell^2}{9\sqrt{3}EI} = 0.06415 \frac{Pa\ell^2}{EI} \\
\Delta_{\text{max}} \text{ (for overhang at } x_1 = a) = \frac{Pa^2}{3EI}(\ell + a) \\
\Delta_s \text{ (between supports)} = \frac{Pax}{6EI}(\ell^2 - x^2) \\
\Delta_{x_1} \text{ (for overhang)} = \frac{Px_1}{6EI} (2a\ell + 3ax_1 - x_1^2)
\]
**Figure 21**  Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

\[ R_1 = V_1 \text{ (max when } a < b) \ldots = \frac{Pb}{\ell} \]
\[ R_2 = V_2 \text{ (max when } a > b) \ldots = \frac{Pa}{\ell} \]
\[ M_{\text{max}} \text{ (at point of load)} \ldots = \frac{Pab}{\ell} \]
\[ M_x \text{ (when } x < a) \ldots = \frac{Pbx}{\ell} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt[3]{\frac{(a + 2b)^2}{3}} \text{ when } a > b \right) = \frac{Pab(a + 2b)}{27EIl^2} \]
\[ \Delta_x \text{ (at point of load)} \ldots = \frac{Pa^2b^3}{3EI\ell^2} \]
\[ \Delta_x \text{ (when } x < a) \ldots = \frac{Pbx}{6EI\ell^2} (\ell^2 - b^2 - x^2) \]
\[ \Delta_x \text{ (when } x > a) \ldots = \frac{Pa(\ell - x)}{6EI\ell^2} (2\ell x - x^2 - a^2) \]
\[ \Delta_{\text{II}} \ldots = \frac{Pabx}{6EI\ell} (\ell + a) \]

**Figure 22**  Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

\[ R_1 \ldots = \frac{w\ell}{2} (\ell - 2c) \]
\[ R_2 \ldots = \frac{w\ell}{2} (\ell - 2a) \]
\[ V_1 \ldots = wa \]
\[ V_2 \ldots = R_1 - V_1 \]
\[ V_3 \ldots = R_2 - V_4 \]
\[ V_4 \ldots = wc \]
\[ V_{x_1} \ldots = V_1 - wx_1 \]
\[ V_x \text{ (when } x < \ell) \ldots = R_1 - w(a + x_1) \]
\[ V_m \text{ (when } a < c) \ldots = R_2 - wc \]
\[ M_1 \ldots = -\frac{wa^2}{2} \]
\[ M_2 \ldots = -\frac{wc^2}{2} \]
\[ M_3 \ldots = R_1 \left( \frac{R_1}{2w} - a \right) \]
\[ M_x \left( \text{max when } x = \frac{R_1}{w} - a \right) \ldots = R_1 x - \frac{w(a + x)^2}{2} \]
Figure 23  Beam Fixed at Both Ends – Uniformly Distributed Load

\[ R = V \quad \ldots \quad = \frac{w\ell}{2} \]

\[ V_x \quad \ldots \quad = w\left(\frac{\ell}{2} - x\right) \]

\[ M_{\text{max}} \text{ (at ends)} \quad \ldots \quad = \frac{w\ell^2}{12} \]

\[ M_i \text{ (at center)} \quad \ldots \quad = \frac{w\ell^2}{24} \]

\[ M_x \quad \ldots \quad = \frac{w}{12}(6\ell x - \ell^2 - 6x^2) \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \quad = \frac{w\ell^4}{384EI} \]

\[ \Delta_x \quad \ldots \quad = \frac{wx^2}{24EI}(\ell - x)^2 \]

Figure 24  Beam Fixed at Both Ends – Concentrated Load at Center

\[ R = V \quad \ldots \quad = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at center and ends)} \quad \ldots \quad = \frac{P\ell}{8} \]

\[ M_x \left(\text{when } x < \frac{\ell}{2}\right) \quad \ldots \quad = \frac{P}{8}(4x - \ell) \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \quad = \frac{P\ell^3}{192EI} \]

\[ \Delta_x \left(\text{when } x < \frac{\ell}{2}\right) \quad \ldots \quad = \frac{Px^2}{48EI}(3\ell - 4x) \]
Figure 25  Beam Fixed at Both Ends – Concentrated Load at Any Point

\[ R_1 = V_1 \text{ (max when } a < b) \quad \ldots \ldots = \frac{pb^2}{\ell^3} (3a + b) \]

\[ R_2 = V_2 \text{ (max when } a > b) \quad \ldots \ldots = \frac{pa^2}{\ell^3} (a + 3b) \]

\[ M_1 \text{ (max when } a < b) \quad \ldots \ldots = \frac{pab^2}{\ell^4} \]

\[ M_2 \text{ (max when } a > b) \quad \ldots \ldots = \frac{pa^2b}{\ell^4} \]

\[ M_a \text{ (at point of load) } \ldots \ldots = \frac{2pa^2b^2}{\ell^5} \]

\[ M_x \text{ (when } x < a) \quad \ldots \ldots = R_1x - \frac{pab^2}{\ell^5} \]

\[ \Delta_{max} \text{ (when } a > b \text{ at } x = \frac{2a\ell}{3a + b}) \quad \ldots \ldots = \frac{2pa^2b^2}{3EI(3a + b)^2} \]

\[ \Delta_x \text{ (at point of load) } \ldots \ldots = \frac{pa^2b^3}{6EI\ell^5} \]

\[ \Delta_x \text{ (when } x < a) \quad \ldots \ldots = \frac{pb^2x^2}{6EI\ell^5} (3a\ell - 3ax - bx) \]

Figure 26  Continuous Beam – Two Equal Spans – Uniform Load on One Span

\[ R_1 = V_1 \quad \ldots \ldots = \frac{7}{16} w\ell \]

\[ R_2 = V_2 + V_1 \quad \ldots \ldots = \frac{5}{8} w\ell \]

\[ R_3 = V_1 \quad \ldots \ldots = -\frac{1}{16} w\ell \]

\[ V_2 \quad \ldots \ldots = \frac{9}{16} w\ell \]

\[ M_{max} \text{ (at } x = \frac{7}{16} \ell \text{) } \ldots \ldots = \frac{49}{512} w\ell^2 \]

\[ M_i \text{ (at support } R_2 \text{) } \ldots \ldots = \frac{1}{16} w\ell^2 \]

\[ M_x \text{ (when } x < \ell) \quad \ldots \ldots = \frac{wx}{16} (7\ell - 8x) \]
**Figure 27** Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

\[
\begin{align*}
R_1 &= V_1 = \frac{13}{32} P \\
R_2 &= V_2 + V_1 = \frac{11}{16} P \\
R_3 &= V_3 = -\frac{3}{32} P \\
V_2 &= \frac{19}{32} P \\
M_{\text{max}} \text{ (at point of load)} &= \frac{13}{64} P\ell \\
M_1 \text{ (at support } R_2) &= \frac{3}{32} P\ell
\end{align*}
\]

**Figure 28** Continuous Beam – Two Equal Spans – Concentrated Load at Any Point

\[
\begin{align*}
R_1 &= V_1 = \frac{Pb}{4\ell^3}(4\ell^2 - a(\ell + a)) \\
R_2 &= V_2 + V_1 = \frac{Pa}{2\ell^3}(2\ell^2 + b(\ell + a)) \\
R_3 &= V_3 = -\frac{Pab}{4\ell^3}(\ell + a) \\
V_2 &= \frac{Pa}{4\ell^3}(4\ell^2 + b(\ell + a)) \\
M_{\text{max}} \text{ (at point of load)} &= \frac{Pab}{4\ell^3}(4\ell^2 - a(\ell + a)) \\
M_1 \text{ (at support } R_2) &= \frac{Pab}{4\ell^3}(\ell + a)
\end{align*}
\]
Figure 29  Continuous Beam – Two Equal Spans – Uniformly Distributed Load

\[ R_1 = V_1 = R_1 = V_1 = \frac{3wl}{8} \]
\[ R_2 = \frac{10wl}{8} \]
\[ V_2 = V_{\text{max}} = \frac{5wl}{8} \]
\[ M_1 = \frac{wl^2}{8} \]
\[ M_2 \left( \text{at } \frac{3l}{8} \right) = \frac{9wl^2}{128} \]
\[ \Delta_{\text{max}} \text{ (at 0.4215} \ell, \text{ approx. from } R_1 \text{ and } R_3) = \frac{wl^4}{185\ell I} \]

Figure 30  Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

\[ R_1 = V_1 = R_1 = V_1 = \frac{5P}{16} \]
\[ R_2 = 2V_2 = \frac{11P}{8} \]
\[ V_2 = P - R_1 = \frac{11P}{16} \]
\[ V_{\text{max}} = V_2 \]
\[ M_1 = -\frac{3Pl}{16} \]
\[ M_2 = \frac{5P\ell}{32} \]
\[ M_4 \text{ (when } x < a) = R_1x \]
Figure 31  Continuous Beam – Two Unequal Spans – Uniformly Distributed Load

\[ R_1 = \frac{M_1}{\ell_1} + \frac{wL_1}{2} \]

\[ R_2 = wL_1 + wL_2 - R_1 - R_3 \]

\[ R_3 = V_4 = \frac{M_1}{\ell_2} + \frac{wL_1}{2} \]

\[ V_1 = R_1 \]

\[ V_2 = wL_1 - R_1 \]

\[ V_3 = wL_2 - R_3 \]

\[ V_4 = R_3 \]

\[ M_4 = -\frac{wL_1^3 + wL_2^3}{8(\ell_1 + \ell_2)} \]

\[ M_{x1} \left( \text{when } x_1 = \frac{R_1}{w} \right) = R_1x_1 - \frac{wx_1^2}{2} \]

\[ M_{x2} \left( \text{when } x_2 = \frac{R_3}{w} \right) = R_3x_2 - \frac{wx_2^2}{2} \]

Figure 32  Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed

\[ R_1 = \frac{M_1}{\ell_1} + \frac{P_1}{2} \]

\[ R_2 = P_1 + P_2 - R_1 - R_3 \]

\[ R_3 = \frac{M_1}{\ell_2} + \frac{P_2}{2} \]

\[ V_1 = R_1 \]

\[ V_2 = P_1 - R_1 \]

\[ V_3 = P_2 - R_3 \]

\[ V_4 = R_3 \]

\[ M_4 = -\frac{3}{16} \left( \frac{P_1\ell_1^2 + P_2\ell_2^2}{\ell_1 + \ell_2} \right) \]

\[ M_{x1} = R_1a \]

\[ M_{x2} = R_3b \]