BEAM DESIGN FORMULAS
WITH SHEAR AND MOMENT
DIAGRAMS

DESIGN AID No. 6
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AMERICAN WOOD COUNCIL
**Introduction**

Figures 1 through 32 provide a series of shear and moment diagrams with accompanying formulas for design of beams under various static loading conditions.

Shear and moment diagrams and formulas are excerpted from the *Western Woods Use Book*, 4th edition, and are provided herein as a courtesy of Western Wood Products Association.

**Notations Relative to “Shear and Moment Diagrams”**

- \( E \) = modulus of elasticity, psi
- \( I \) = moment of inertia, in.\(^4\)
- \( L \) = span length of the bending member, ft.
- \( l \) = span length of the bending member, in.
- \( M \) = maximum bending moment, in.-lbs.
- \( P \) = total concentrated load, lbs.
- \( R \) = reaction load at bearing point, lbs.
- \( V \) = shear force, lbs.
- \( W \) = total uniform load, lbs.
- \( w \) = load per unit length, lbs./in.
- \( \Delta \) = deflection or deformation, in.
- \( x \) = horizontal distance from reaction to point on beam, in.

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**Figure 1** Simple Beam – Uniformly Distributed Load

\[ R = V = \frac{w\ell}{2} \]

\[ V_x = w\left(\frac{\ell}{2} - x\right) \]

\[ M_{max} \text{ (at center)} = \frac{w\ell^2}{8} \]

\[ M_x = \frac{ux}{2} (\ell - x) \]

\[ \Delta_{max} \text{ (at center)} = \frac{5w\ell^4}{384 EI} \]

\[ \Delta_x = \frac{ux}{24 EI} (\ell^3 - 2\ell x^2 + x^3) \]

---

**Figure 2** Simple Beam – Uniform Load Partially Distributed

\[ R_1 = V_1 \text{ (max when } a < c \text{)} = \frac{wb}{2\ell} (2c + b) \]

\[ R_2 = V_2 \text{ (max when } a > c \text{)} = \frac{wb}{2\ell} (2a + b) \]

\[ V_x \text{ (when } x > a \text{ and } < (a + b) \text{)} = R_1 - w(x - a) \]

\[ M_{max} \text{ (at } x = a + \frac{R_1}{w} \text{)} = R_1 \left(a + \frac{R_1}{2w}\right) \]

\[ M_x \text{ (when } x < a \text{)} = R_1 x \]

\[ M_x \text{ (when } x > a \text{ and } < (a + b) \text{)} = R_1 x - \frac{w}{2} (x - a)^2 \]

\[ M_x \text{ (when } x > (a + b) \text{)} = R_2 (\ell - x) \]
Figure 3  Simple Beam – Uniform Load Partially Distributed at One End

\[ R_1 = V_1 = \frac{wa}{2\ell} (2\ell - a) \]
\[ R_2 = V_2 = \frac{wa^2}{2\ell} \]
\[ V_x \quad \text{(when } x < a) \quad . . . . . = R_1 - wx \]
\[ M_{\max} \left( \text{at } x = \frac{R_1}{w} \right) \quad . . . . . = \frac{R_1^2}{2w} \]
\[ M_x \quad \text{(when } x < a) \quad . . . . . = R_1 x - \frac{wx^2}{2} \]
\[ M_x \quad \text{(when } x > a) \quad . . . . . = R_2 (\ell - x) \]
\[ \Delta_x \quad \text{(when } x < a) \quad . . . . . = \frac{wx}{24 E I \ell} \left( a^2 (2\ell - a)^2 - 2ax (2\ell - a) + \ell x^3 \right) \]
\[ \Delta_x \quad \text{(when } x > a) \quad . . . . . = \frac{wa^2(\ell - x)}{24 E I \ell} (4x\ell - 2x^2 - a^2) \]

Figure 4  Simple Beam – Uniform Load Partially Distributed at Each End

\[ R_1 = V_1 = \frac{w_1 a (2\ell - a) + w_2 c^2}{2\ell} \]
\[ R_2 = V_2 = \frac{w_2 c (2\ell - c) + w_1 a^2}{2\ell} \]
\[ V_x \quad \text{(when } x < a) \quad . . . . . = R_1 - w_1 x \]
\[ V_x \quad \text{(when } x > a \text{ and } < (a + b)) \quad . . . . . = R_1 - w_1 a \]
\[ V_x \quad \text{(when } x > (a + b)) \quad . . . . . = R_2 - w_2 (\ell - x) \]
\[ M_{\max} \left( \text{at } x = \frac{R_1}{w_1} \text{ when } R_1 < w_1 a \right) \quad . . . . . = \frac{R_1^2}{2w_1} \]
\[ M_{\max} \left( \text{at } x = \ell - \frac{R_2}{w_2} \text{ when } R_2 < w_2 c \right) \quad = \frac{R_2^2}{2w_2} \]
\[ M_x \quad \text{(when } x < a) \quad . . . . . = R_1 x - \frac{w_1 x^2}{2} \]
\[ M_x \quad \text{(when } x > a \text{ and } < (a + b)) \quad . . . . . = R_1 x - \frac{w_1 a}{2} (2x - a) \]
\[ M_x \quad \text{(when } x > (a + b)) \quad . . . . . = R_2 (\ell - x) - \frac{w_2 (\ell - x)^2}{2} \]
Figure 5  Simple Beam – Load Increasing Uniformly to One End

\[ R_1 = \frac{V_1}{3} = \frac{W}{3} \]
\[ R_2 = \frac{V_2}{3} = \frac{2W}{3} \]
\[ V_x = \frac{W}{3} - \frac{Wx^2}{\ell^2} \]
\[ M_{\text{max}} \left( \text{at } x = \frac{\ell}{\sqrt{3}} = 0.5774 \ell \right) = \frac{2W\ell}{9\sqrt{3}} = 0.1283W\ell \]
\[ M_x \left( \text{at } x = \ell \right) = \frac{Wx}{3\ell^2} \left( \ell^2 - x^2 \right) \]
\[ \Delta_{\text{max}} \left( \text{at } x = \ell \right) = \frac{W \ell^3}{180EI \ell^2} \left( 3x^4 - 10\ell^2 x^2 + 7\ell^4 \right) \]

Figure 6  Simple Beam – Load Increasing Uniformly to Center

\[ R = \frac{W}{2} \]
\[ V_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{W}{2\ell^2} \left( \ell^2 - 4x^2 \right) \]
\[ M_{\text{max}} \left( \text{at center} \right) = \frac{W\ell}{6} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) = \frac{Wx}{3\ell^2} \left( 1 - \frac{2x^2}{3\ell^2} \right) \]
\[ \Delta_{\text{max}} \left( \text{at center} \right) = \frac{W \ell^3}{60EI} \]
\[ \Delta_x \left( \text{at center} \right) = \frac{Wx}{480EI \ell^2} \left( 5\ell^2 - 4x^2 \right)^2 \]
**Figure 7** Simple Beam – Concentrated Load at Center

\[ R = V \quad \ldots \quad = \frac{P}{2} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \quad = \frac{P\ell}{4} \]
\[ M_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \quad = \frac{Px}{2} \]
\[ \Delta_{\text{max}} \text{ (at point of load)} \quad \ldots \quad = \frac{P\ell^3}{48EI} \]
\[ \Delta_x \left( \text{when } x < \frac{\ell}{2} \right) \quad \ldots \quad = \frac{Px}{48EI} (3\ell^2 - 4x^2) \]

**Figure 8** Simple Beam – Concentrated Load at Any Point

\[ R_1 = V_1 \; \text{ (max when } a < b \) \quad \ldots \quad = \frac{Pb}{\ell} \]
\[ R_2 = V_2 \; \text{ (max when } a > b \) \quad \ldots \quad = \frac{Pa}{\ell} \]
\[ M_{\text{max}} \text{ (at point of load)} \quad \ldots \quad = \frac{Pab}{\ell} \]
\[ M_x \left( \text{when } x < a \right) \quad \ldots \quad = \frac{Pbx}{\ell} \]
\[ \Delta_{\text{max}} \left( \text{at } x = \sqrt{\frac{a(a + 2b)}{3}} \text{ when } a > b \right) \quad \ldots \quad = \frac{Pab(a + 2b)\sqrt{3(a + 2b)}}{27EIl} \]
\[ \Delta_x \text{ (at point of load)} \quad \ldots \quad = \frac{Pa^2b^2}{3EIl} \]
\[ \Delta_x \left( \text{when } x < a \right) \quad \ldots \quad = \frac{Pbx}{6EIl} (\ell^2 - b^2 - x^2) \]
\[ \Delta_x \left( \text{when } x > a \right) \quad \ldots \quad = \frac{Pa(\ell - x)}{6EIl} (2\ell x - x^2 - a^2) \]
Figure 9  Simple Beam – Two Equal Concentrated Loads Symmetrically Placed

\[ R = V \quad \ldots \quad = P \]

\[ M_{\text{max}} \text{ (between loads)} \quad \ldots \quad = Pa \]

\[ M_{\varepsilon} \text{ (when } x < a \text{)} \quad \ldots \quad = Px \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \ldots \quad = \frac{Pa}{24EI} (3\ell^2 - 4a^2) \]

\[ \Delta_{\varepsilon} \text{ (when } x < a \text{)} \quad \ldots \quad = \frac{Px}{6EI} (3\ell a - 3a^2 - x^2) \]

\[ \Delta_{\varepsilon} \text{ (when } x > a \text{ and } < (\ell - a) \text{)} \quad \ldots \quad = \frac{Pa}{6EI} (3\ell x - 3x^2 - a^2) \]

Figure 10  Simple Beam – Two Equal Concentrated Loads Unsymmetrically Placed

\[ R_1 = V_1 \text{ (max when } a < b \text{)} \quad \ldots \quad = \frac{P}{\ell} (\ell - a + b) \]

\[ R_2 = V_2 \text{ (max when } a > b \text{)} \quad \ldots \quad = \frac{P}{\ell} (\ell - b + a) \]

\[ V_\varepsilon \text{ (when } x > a \text{ and } < (\ell - b) \text{)} \quad \ldots \quad = \frac{P}{\ell} (b - a) \]

\[ M_1 \text{ (max when } a > b \text{)} \quad \ldots \quad = R_1 a \]

\[ M_2 \text{ (max when } a < b \text{)} \quad \ldots \quad = R_2 b \]

\[ M_{\varepsilon} \text{ (when } x < a \text{)} \quad \ldots \quad = R_1 x \]

\[ M_{\varepsilon} \text{ (when } x > a \text{ and } < (\ell - b) \text{)} \quad \ldots \quad = R_1 x - P(x - a) \]
**Figure 11**  
**Simple Beam – Two Unequal Concentrated Loads Unsymmetrically Placed**

\[ R_1 = V_1 = \frac{P_1(\ell - a) + P_2b}{\ell} \]

\[ R_2 = V_2 = \frac{P_1a + P_2(\ell - b)}{\ell} \]

\[ V_x \quad \text{(when} \ x > a \ \text{and} \ (\ell - b) \ \text{)} \quad = \quad R_1 - P_1 \]

\[ M_1 \quad \text{(max when} \ R_1 < P_1) \quad = \quad R_1a \]

\[ M_2 \quad \text{(max when} \ R_2 < P_2) \quad = \quad R_2b \]

\[ M_x \quad \text{(when} \ x < a) \quad = \quad R_1x \]

\[ M_x \quad \text{(when} \ x > a \ \text{and} \ (\ell - b) \ \text{)} \quad = \quad R_1x - P_1(x - a) \]

**Figure 12**  
**Cantilever Beam – Uniformly Distributed Load**

\[ R = V = w\ell \]

\[ V_x = wx \]

\[ M_{\text{max}} \quad \text{(at fixed end)} \quad \frac{w\ell^2}{2} \]

\[ M_x = \frac{wx^2}{2} \]

\[ \Delta_{\text{max}} \quad \text{(at free end)} \quad \frac{w\ell^4}{8EI} \]

\[ \Delta_x = \frac{w}{24EI} (x^4 - 4\ell^3x + 3\ell^4) \]
**Figure 13** Cantilever Beam – Concentrated Load at Free End

\[ R = V = P \]
\[ M_{\text{max}} \text{ (at fixed end)} = P\ell \]
\[ M_x = Px \]
\[ \Delta_{\text{max}} \text{ (at free end)} = \frac{P\ell^3}{3EI} \]
\[ \Delta_x = \frac{P}{6EI}(2\ell^3 - 3\ell^2x + x^3) \]

**Figure 14** Cantilever Beam – Concentrated Load at Any Point

\[ R = V = P \]
\[ M_{\text{max}} \text{ (at fixed end)} = Pb \]
\[ M_x \text{ (when } x > a) = P(x - a) \]
\[ \Delta_{\text{max}} \text{ (at free end)} = \frac{Pb^2}{6EI}(3\ell - b) \]
\[ \Delta_x \text{ (at point of load)} = \frac{Pb^3}{3EI} \]
\[ \Delta_x \text{ (when } x < a) = \frac{Pb^2}{6EI}(3\ell - 3x - b) \]
\[ \Delta_x \text{ (when } x > a) = \frac{P(\ell - x)^2}{6EI}(3b - \ell + x) \]
Figure 15  Beam Fixed at One End, Supported at Other – Uniformly Distributed Load

\[ R_1 = V_1 = \frac{3w\ell}{8} \]
\[ R_2 = V_2 = \frac{5w\ell}{8} \]
\[ V_x = R_1 - wx \]
\[ M_{\text{max}} = \frac{w\ell^2}{8} \]
\[ M_1 \left( \text{at} \ x = \frac{3\ell}{8} \right) = \frac{9w\ell^2}{128} \]
\[ M_x = R_1x - \frac{wx^2}{2} \]
\[ \Delta_{\text{max}} \left( \text{at} \ x = \frac{\ell}{16}(1 + \sqrt{33}) = .4215\ell \right) = \frac{w\ell^4}{185EI} \]
\[ \Delta_x = \frac{wx}{48EI}(\ell^3 - 3\ell x^2 + 2x^3) \]

Figure 16  Beam Fixed at One End, Supported at Other – Concentrated Load at Center

\[ R_1 = V_1 = \frac{5P}{16} \]
\[ R_2 = V_2 = \frac{11P}{16} \]
\[ M_{\text{max}} \left( \text{at} \ \text{fixed end} \right) = \frac{3P\ell}{16} \]
\[ M_1 \left( \text{at} \ \text{point of load} \right) = \frac{5P\ell}{32} \]
\[ M_x \left( \text{when} \ x < \frac{\ell}{2} \right) = \frac{5Px}{16} \]
\[ M_x \left( \text{when} \ x > \frac{\ell}{2} \right) = P \left( \frac{\ell}{2} - \frac{11x}{16} \right) \]
\[ \Delta_{\text{max}} \left( \text{at} \ x = \frac{\ell}{15} = .4472\ell \right) = \frac{P\ell^3}{48EI\sqrt{5}} = .009317\frac{P\ell^3}{EI} \]
\[ \Delta_x \left( \text{at} \ \text{point of load} \right) = \frac{7P\ell^3}{768EI} \]
\[ \Delta_x \left( \text{when} \ x < \frac{\ell}{2} \right) = \frac{Px}{96EI} (3\ell^2 - 5x^2) \]
\[ \Delta_x \left( \text{when} \ x > \frac{\ell}{2} \right) = \frac{P}{96EI} (x - \ell)^2(11x - 2\ell) \]
Figure 17  Beam Fixed at One End, Supported at Other – Concentrated Load at Any Point

\[ R_1 = V_1 \quad \ldots \ldots \ldots \ldots \ldots = \frac{Pb^2}{2\ell^3} (a + 2\ell) \]
\[ R_2 = V_2 \quad \ldots \ldots \ldots \ldots \ldots = \frac{Pa}{2\ell^3} (3\ell^2 - a^2) \]
\[ M_1 \text{ (at point of load)} \quad \ldots \ldots \ldots \ldots = R_1 a \]
\[ M_2 \text{ (at fixed end)} \quad \ldots \ldots \ldots \ldots = \frac{Pab}{2\ell^3} (a + \ell) \]
\[ M_s \text{ (when } x < a) \quad \ldots \ldots \ldots \ldots = R_1 x \]
\[ M_s \text{ (when } x > a) \quad \ldots \ldots \ldots \ldots = R_1 x - P(x - a) \]
\[ \Delta_{\text{max}} \text{ (when } a < 0.414\ell \text{ at } x = \ell - \frac{\ell^2 + a^2}{3\ell^2 - a^2} \ldots = \frac{Pb}{3EI} \frac{(\ell^2 - a^2)^3}{(3\ell^2 - a^2)^2} \]
\[ \Delta_{\text{max}} \text{ (when } a > 0.414\ell \text{ at } x = \ell \frac{a}{2\ell + a} \ldots = \frac{Pb}{6EI} \frac{a}{2\ell + a} \]
\[ \Delta_s \text{ (at point of load)} \ldots \ldots \ldots \ldots = \frac{Pa}{12EI\ell^3} (3\ell + a) \]
\[ \Delta_s \text{ (when } x < a) \ldots \ldots \ldots \ldots = \frac{Pb^2}{12EI\ell^3} (3a\ell^2 - 2ax^2 - ax^2) \]
\[ \Delta_s \text{ (when } x > a) \ldots \ldots \ldots \ldots = \frac{Pa}{12EI\ell^3} (\ell - x)^2(3\ell^2 x - a^2 x - 2a^2 \ell) \]

Figure 18  Beam Overhanging One Support – Uniformly Distributed Load

\[ R_1 = V_1 \quad \ldots \ldots \ldots \ldots = \frac{w}{2\ell} (\ell^2 - a^2) \]
\[ R_2 = V_2 + V_1 \quad \ldots \ldots \ldots \ldots = \frac{w}{2\ell} (\ell + a)^2 \]
\[ V_2 \quad \ldots \ldots \ldots \ldots = wa \]
\[ V_3 \quad \ldots \ldots \ldots \ldots = \frac{w}{2\ell} (\ell^2 + a^2) \]
\[ V_s \text{ (between supports)} \ldots \ldots \ldots = R_1 - w x \]
\[ V_{s_1} \text{ (for overhang)} \ldots \ldots \ldots = w(a - x) \]
\[ M_1 \text{ (at } x = \frac{\ell}{2} \left(1 - \frac{a^2}{\ell^2}\right) \ldots \ldots \ldots = \frac{w}{8\ell^2} (\ell + a)^2(\ell - a)^2 \]
\[ M_2 \text{ (at } R_2) \ldots \ldots \ldots \ldots = \frac{wa^2}{2} \]
\[ M_s \text{ (between supports)} \ldots \ldots \ldots = \frac{wx}{2\ell} (\ell^2 - a^2 - x\ell) \]
\[ M_{s_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{w}{2} (a - x)^3 \]
\[ \Delta_s \text{ (between supports)} \ldots \ldots \ldots = \frac{wx}{24EI\ell} (\ell^4 - 2\ell^2 x^2 + \ell x^3 - 2a^2 \ell^2 + 2a^2 x^2) \]
\[ \Delta_{s_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{wx}{24EI} (4a^2 \ell - \ell^3 + 6a^2 x_1 - 4ax_1^2 + x_1^3) \]
Figure 19  Beam Overhanging One Support – Uniformly Distributed Load on Overhang

\[ R_1 = V_1 \ldots \ldots \ldots \ldots \ldots = \frac{wa^2}{2\ell} \]
\[ R_2 = V_1 + V_2 \ldots \ldots \ldots \ldots \ldots = \frac{wa(2\ell + a)}{2\ell} \]
\[ V_2 \ldots \ldots \ldots \ldots \ldots = wa \]
\[ V_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = wa(a - x_1) \]
\[ M_{\text{max}} \text{ (at } R_1) \ldots \ldots \ldots = \frac{wa^2}{2\ell} \]
\[ M_x \text{ (between supports)} \ldots \ldots \ldots = \frac{wa^2 \ell}{2\ell} \]
\[ M_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{w}{2}(a - x_1)^2 \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{wa^2 \ell^3}{18\sqrt{3}EI} = .03208 \frac{wa^2 \ell^3}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) \ldots \ldots = \frac{wa^3}{24EI} (4\ell + 3a) \]
\[ \Delta_x \text{ (between supports)} \ldots \ldots \ldots = \frac{wa^2 \ell}{12EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{wa_1^3}{24EI} (4a^2 \ell + 6a^2 x_1 - 4ax_1^2 + x_1^3) \]

Figure 20  Beam Overhanging One Support – Concentrated Load at End of Overhang

\[ R_1 = V_1 \ldots \ldots \ldots \ldots \ldots = \frac{Pa}{\ell} \]
\[ R_2 = V_1 + V_2 \ldots \ldots \ldots \ldots \ldots = \frac{P}{\ell} (\ell + a) \]
\[ V_2 \ldots \ldots \ldots \ldots \ldots = P \]
\[ M_{\text{max}} \text{ (at } R_1) \ldots \ldots \ldots = Pa \]
\[ M_x \text{ (between supports)} \ldots \ldots \ldots = \frac{Pa \ell}{\ell} \]
\[ M_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = P(a - x_1) \]
\[ \Delta_{\text{max}} \left( \text{between supports at } x = \frac{\ell}{\sqrt{3}} \right) = \frac{Pa \ell^3}{9\sqrt{3}EI} = .06415 \frac{Pa \ell^3}{EI} \]
\[ \Delta_{\text{max}} \text{ (for overhang at } x_1 = a) \ldots \ldots = \frac{Pa^2}{3EI} (\ell + a) \]
\[ \Delta_x \text{ (between supports)} \ldots \ldots \ldots = \frac{Pa \ell}{6EI} (\ell^2 - x^2) \]
\[ \Delta_{x_1} \text{ (for overhang)} \ldots \ldots \ldots = \frac{P_1}{6EI} (2a\ell + 3ax_1 - x_1^2) \]
Figure 21  Beam Overhanging One Support – Concentrated Load at Any Point Between Supports

\[
R_1 = V_1 \text{ (max when } a < b) \quad \cdots \quad = \frac{Pb}{\ell} \\
R_2 = V_2 \text{ (max when } a > b) \quad \cdots \quad = \frac{Pa}{\ell} \\
M_{\text{max}} \text{ (at point of load)} \quad \cdots \quad = \frac{Pab}{\ell} \\
M_x \text{ (when } x < a) \quad \cdots \quad = \frac{Pbx}{\ell} \\
\Delta_{\text{max}} \left( \text{ at } x = \frac{a(a+2b)^2}{3} \text{ when } a > b \right) \quad = \frac{P(b+2b+3a+2b^2)}{27EI\ell} \\
\Delta_a \text{ (at point of load)} \quad \cdots \quad = \frac{Pa^2b^2}{3EI\ell} \\
\Delta_x \text{ (when } x < a) \quad \cdots \quad = \frac{Pbx}{6EI\ell} \left( \ell^2 - b^2 - x^2 \right) \\
\Delta_x \text{ (when } x > a) \quad \cdots \quad = \frac{Pa(\ell-x)}{6EI\ell} \left( 2\ell x - x^2 - a^2 \right) \\
\Delta_{\text{ini}} \quad \cdots \quad = \frac{Pabx_{1+}}{6EI\ell} (\ell + a)
\]

Figure 22  Beam Overhanging Both Supports – Unequal Overhangs – Uniformly Distributed Load

\[
R_1 \quad \cdots \quad = \frac{w\ell}{2} (\ell - 2c) \\
R_2 \quad \cdots \quad = \frac{w\ell}{2} (\ell - 2a) \\
V_1 \quad \cdots \quad = wa \\
V_2 \quad \cdots \quad = R_1 - V_1 \\
V_3 \quad \cdots \quad = R_2 - V_1 \\
V_4 \quad \cdots \quad = wc \\
V_{x_1} \quad \cdots \quad = V_1 - wx_1 \\
V_x \text{ (when } x < \ell) \quad \cdots \quad = R_1 - w(a + x_1) \\
V_m \text{ (when } a < c) \quad \cdots \quad = R_2 - wc \\
M_1 \quad \cdots \quad = -\frac{wa^2}{2} \\
M_2 \quad \cdots \quad = -\frac{wc^2}{2} \\
M_3 \quad \cdots \quad = R_1 \left( R_2 - a \right) \\
M_x \left( \text{max when } x = \frac{R_1}{w} - a \right) \quad \cdots \quad = R_1 x - \frac{w(a + x)^2}{2}
\]
Figure 23  Beam Fixed at Both Ends – Uniformly Distributed Load

\[ R = V \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{w\ell}{2} \]

\[ V_x \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = w\left(\frac{\ell}{2} - x\right) \]

\[ M_{\text{max}} \text{ (at ends)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{w\ell^2}{12} \]

\[ M_i \text{ (at center)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{w\ell^2}{24} \]

\[ M_x \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{w(6\ell x - \ell^2 - 6x^2)}{12} \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{w\ell^4}{384EI} \]

\[ \Delta_x \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{wx^2}{24EI} (\ell - x)^2 \]

Figure 24  Beam Fixed at Both Ends – Concentrated Load at Center

\[ R = V \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{P}{2} \]

\[ M_{\text{max}} \text{ (at center and ends)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{P\ell}{8} \]

\[ M_x \left(\text{when } x < \frac{\ell}{2}\right) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{P}{8}(4x - \ell) \]

\[ \Delta_{\text{max}} \text{ (at center)} \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{P\ell^3}{192EI} \]

\[ \Delta_x \left(\text{when } x < \frac{\ell}{2}\right) \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots = \frac{Px^2}{48EI} (3\ell - 4x) \]
Figure 25  Beam Fixed at Both Ends – Concentrated Load at Any Point

\[ R_1 = V_1 \quad \text{max when } a < b \quad \therefore \quad = \frac{Pb^2}{\ell^3} (3a + b) \]
\[ R_2 = V_2 \quad \text{max when } a > b \quad \therefore \quad = \frac{Pb^2}{\ell^3} (a + 3b) \]
\[ M_1 \quad \text{max when } a < b \quad \therefore \quad = \frac{Pab^2}{\ell^2} \]
\[ M_2 \quad \text{max when } a > b \quad \therefore \quad = \frac{Pab^2}{\ell^2} \]
\[ M_3 \quad \text{at point of load} \quad \therefore \quad = \frac{2Pa^3b^3}{\ell^3} \]
\[ M_4 \quad \text{when } x < a \quad \therefore \quad = R_1x - \frac{Pab^2}{\ell^2} \]
\[ \Delta_{\text{max}} \quad \text{when } a > b \text{ at } x = \frac{2 \alpha}{3a + b} \quad \therefore \quad = \frac{2Pa^3b^3}{3EI(3a + b)^2} \]
\[ \Delta_4 \quad \text{at point of load} \quad \therefore \quad = \frac{Pa^3b^3}{3EI^3} \]
\[ \Delta_4 \quad \text{when } x < a \quad \therefore \quad = \frac{Pb^3x^2}{6EI^3} (3a - 3ax - bx) \]

Figure 26  Continuous Beam – Two Equal Spans – Uniform Load on One Span

\[ R_1 = V_1 \quad \therefore \quad = \frac{7}{16} w\ell \]
\[ R_2 = V_2 + V_1 \quad \therefore \quad = \frac{5}{8} w\ell \]
\[ R_3 = V_1 \quad \therefore \quad = -\frac{1}{16} w\ell \]
\[ V_2 \quad \therefore \quad = \frac{9}{16} w\ell \]
\[ M_{\text{max}} \quad \text{at } x = \frac{7}{16} \ell \quad \therefore \quad = \frac{49}{512} w\ell^2 \]
\[ M_1 \quad \text{at support } R_1 \quad \therefore \quad = \frac{1}{16} w\ell^2 \]
\[ M_2 \quad \text{when } x < \ell \quad \therefore \quad = \frac{wx}{16} (7\ell - 8x) \]
Figure 27  Continuous Beam – Two Equal Spans – Concentrated Load at Center of One Span

\[ R_1 = V_1 = \frac{13}{32} P \]
\[ R_2 = V_2 + V_1 = \frac{11}{16} P \]
\[ R_3 = V_3 = -\frac{3}{32} P \]
\[ V_2 = \frac{19}{32} P \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{13}{64} P\ell \]
\[ M_i \text{ (at support } R_2) = \frac{3}{32} P\ell \]

Figure 28  Continuous Beam – Two Equal Spans – Concentrated Load at Any Point

\[ R_1 = V_1 = \frac{Pb}{4\ell^3} (4\ell^2 - a(\ell + a)) \]
\[ R_2 = V_2 + V_1 = \frac{Pa}{2\ell^3} (2\ell^2 + b(\ell + a)) \]
\[ R_3 = V_3 = -\frac{Pab}{4\ell^3} (\ell + a) \]
\[ V_2 = \frac{Pb}{4\ell^3} (4\ell^2 + b(\ell + a)) \]
\[ M_{\text{max}} \text{ (at point of load)} = \frac{Pab}{4\ell^3} (4\ell^2 - a(\ell + a)) \]
\[ M_i \text{ (at support } R_2) = \frac{Pab}{4\ell^3} (\ell + a) \]
Figure 29  Continuous Beam – Two Equal Spans – Uniformly Distributed Load

\[ R_1 = V_1 = R_3 = V_3 = \frac{3w\ell}{8} \]
\[ R_2 = \frac{10w\ell}{8} \]
\[ V_2 = V_{\text{max}} = \frac{5w\ell}{8} \]
\[ M_1 = \frac{w\ell^2}{8} \]
\[ M_2 \left( \text{at } \frac{3\ell}{8} \right) = \frac{9w\ell^2}{128} \]
\[ \Delta_{\text{max}} (\text{at } 0.4215 \ell, \text{approx. from } R_1 \text{ and } R_3) = \frac{w\ell^4}{185EI} \]

Figure 30  Continuous Beam – Two Equal Spans – Two Equal Concentrated Loads Symmetrically Placed

\[ R_1 = V_1 = R_2 = V_3 = \frac{5P}{16} \]
\[ R_2 = 2V_2 = \frac{11P}{8} \]
\[ V_2 = P - R_1 = \frac{11P}{16} \]
\[ V_{\text{max}} = V_2 \]
\[ M_1 = -\frac{3P\ell}{16} \]
\[ M_2 = \frac{5P\ell}{32} \]
\[ M_x (\text{when } x < a) = R_1x \]
Figure 31  Continuous Beam – Two Unequal Spans – Uniformly Distributed Load

\[ R_1 = \frac{M_1}{\ell_1} + \frac{w\ell_1}{2} \]
\[ R_2 = w\ell_1 + w\ell_2 - R_1 - R_3 \]
\[ R_3 = V_4 = \frac{M_1}{\ell_2} + \frac{w\ell_2}{2} \]
\[ V_1 = R_1 - V_4 \]
\[ V_2 = w\ell_2 - R_3 \]
\[ M_4 = -\frac{w\ell_1^3 + w\ell_2^3}{8(\ell_1 + \ell_2)} \]
\[ M_{x_1} \left( \text{when } x_1 = \frac{R_1}{w} \right) = R_1x_1 - \frac{wx_1^2}{2} \]
\[ M_{x_2} \left( \text{when } x_2 = \frac{R_3}{w} \right) = R_3x_2 - \frac{wx_2^2}{2} \]

Figure 32  Continuous Beam – Two Unequal Spans – Concentrated Load on Each Span Symmetrically Placed

\[ R_1 = \frac{M_1}{\ell_1} + \frac{P_1}{2} \]
\[ R_2 = P_1 + P_2 - R_1 - R_3 \]
\[ R_3 = \frac{M_1}{\ell_2} + \frac{P_2}{2} \]
\[ V_1 = R_1 \]
\[ V_2 = P_1 - R_1 \]
\[ V_3 = P_2 - R_3 \]
\[ V_4 = R_3 \]
\[ M_4 = -\frac{3}{16} \left( \frac{P_1\ell_1^2 + P_2\ell_2^2}{\ell_1 + \ell_2} \right) \]
\[ M_{x_1} = R_1a \]
\[ M_{x_2} = R_2b \]